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Supersymmetry of the photon

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Abstract. The supersymmetric massless states of a relativistic extension of Witten's supersymmetric quantum mechanics are shown to correspond to Abelian gauge fields realised by antisymmetric tensors. The massive states, which are only partially supersymmetric, include spin-1 and spin-0 particles. In particular the Maxwell and Proca fields are obtained from $N=2$ supersymmetric relativistic mechanics by first quantisation in much the same way as the Dirac theory arises in the $N=1$ case. The particles may be coupled supersymmetrically to external scalar and complex Hermitian tensor fields. The latter reduce in special cases to the Riemannian metric of external gravitation and the Kähler metric implied by a vector field coupling. All the couplings exhibit a quadrupole characteristic of the particles. In particular supersymmetry requires their electric charge and magnetic dipole moment to vanish, and there is no coupling to torsion in Riemann-Cartan spacetime. Quantisation in external fields yields covariant generalisations of the classical tensor field equations. Finally classical equations of translational and spin motion involving only real quantities are obtained from the Heisenberg equations.

1. Introduction

The status of spacetime (Poincaré) supersymmetry as a fundamental symmetry of the physical world is still unclear. Despite the unique mathematical virtues of the superstring theories they can be reconciled with experiment only if it is assumed that supersymmetry is broken at some stage. As a matter of fact, considerable effort has been devoted to the study of spontaneous supersymmetry breaking. The complexity of the problem prompted the invention of model theories, among which Witten's supersymmetric quantum mechanics (Witten 1981) figures most prominently. This model and its higher-dimensional generalisations are interesting also from the purely mathematical point of view owing to the existence of a new topological invariant, the Witten index (Witten 1982), which allows concise derivation of the Atiyah-Singer index theorem in various circumstances (Alvarez-Gaumé 1983). Another active area of research in these models has been the associated Nicolai maps (Nicolai 1980a, b) and their fermion sector structure (Claudson and Halpern 1986, Graham and Roekaerts 1986).

The Witten model lies also at the basis of the present paper. However our objective is quite different from that of the works mentioned above. The relativistic supersymmetric quantum mechanics expounded below is not considered as a 'model' of field theory in $(0+1)$ dimensions but rather as the correct description of some fundamental particles of nature at the first quantised level. Supersymmetry with respect to the affine relativistic evolution parameter, henceforth called proper-time supersymmetry, is the main ingredient of this approach. It is not a widely appreciated fact that proper-time supersymmetry (to be distinguished from spacetime supersymmetry) is indeed realised in nature as exemplified by the Dirac particle (DiVecchia and Ravndal 1979, Ravndal 1980, Rumpf 1982, 1986a). The Dirac equation is the supersymmetry condition on the states of the quantum theory implied by the pseudoclassical Lagrangian

$$L=\tfrac{1}{2}(\dot{x}^2-\mathrm{i}\xi\dot{\xi}). \tag{1.1}$$

This Lagrangian exhibits a global simple supersymmetry involving the position coordinate x and the real anticommuting 4-vector ξ. Upon quantisation, the components ξ^a become essentially the Dirac matrices γ^a. (It should be noted that there exist supersymmetric Lagrangians different from (1.1) yielding the Dirac particle dynamics. Historically the first such Lagrangian proposed was an extension of the reparametrisation-invariant scalar particle Lagrangian $L=(\dot{x}^2)^{1/2}$ and possessed even *local* supersymmetry (Berezin and Marinov 1977). The affine parametrisation implied by (1.1) yields by far the simplest theory, however.)

The starting point of the present paper is the $N=2$ supersymmetric extension of the Lagrangian (1.1)

$$L=\tfrac{1}{2}\dot{x}^2-\mathrm{i}\xi^*\dot{\xi} \tag{1.2}$$

where the ξ^a are odd elements of a Grassmann algebra with involution (denoted by an asterisk). In one dimension (1.2) is the basis of the Witten model. If x takes values not in spacetime, but some 'internal' Riemannian manifold, (1.2) is the prototype of the $N=2$ supersymmetric non-linear σ model in $(0+1)$ dimensions (the reader should be cautioned that the same model is also called $N=1$ supersymmetric by some authors). Of course in both cases one obtains a non-trivial theory only after the introduction of supersymmetric couplings. For the relativistic system one suspects that quantisation will yield the classical field equations describing spin-1 particles, as the Lagrangian (1.2) involves twice as many 'spin variables' as (1.1). Interesting questions then arise. What are the external field couplings compatible with proper-time supersymmetry, what is their physical interpretation, and what information about the classical limit of the particle dynamics can be gained from the supersymmetric formalism? This paper aims at a thorough discussion of these issues.

In § 2 of this paper the relativistic generalisation of the Witten model in the free-particle case is introduced. It turns out that the wavefunctions are inhomogeneous differential forms (or antisymmetric tensor fields) obeying either a Lorentz or Bianchi type condition. Each condition implies partial supersymmetry of the states. Fully $N=2$ supersymmetric states exist only in the case of vanishing mass and correspond to Abelian gauge fields. The states of definite 'fermion' number correspond to particles of spin 0 and spin 1 (helicities 0 and 1 in the massless case). In the subsequent sections all possible supersymmetric couplings to external fields are studied. The simple case of an external scalar field treated in § 3 serves to demonstrate the consistency of the quantisation condition involving one supersymmetry generator under quite general circumstances. Section 4 is devoted to the external gravitational field. The coupling to an external vector field (tentatively identified with the Maxwell field) investigated in § 5 turns out to be rather involved. The electric charge and magnetic dipole moment of the particle have to vanish, but it may have a non-trivial electric quadrupole moment. The coupling can be described in purely geometrical terms upon the introduction of a complex Kähler metric involving the field strength of the vector field. Both the gravitational and vector field coupling are special cases of an external complex Hermitian metric which is the subject of § 6. We derive the classical field equations in an external field from a variational principle (whose existence is a consequence of supersymmetry) and discuss the classical limit of the Heisenberg equations of motion for the position and spin observables. For the proper identification of this limit it will be helpful to let Planck's constant $\hbar$ appear explicitly in all equations. In the final section we assess the physical relevance of our results, point out a remarkable confirmation of them in quantum field theory and propose possible areas for future investigation. An appendix serves to illustrate the complications that result from the minimal electromagnetic coupling of vector fields.

2. Relativistic $N=2$ supersymmetric quantum mechanics: the free particle

The relativistic generalisation of the Witten model (Witten 1981) is straightforward if the time coordinate x^0 is treated on equal footing with the space coordinates and an affine parameter s (which in the massive case becomes τ/m, τ being the proper time) is introduced. The coordinates $x^a(s)$, $\xi^a(s)$, $\xi^{*a}(s)$ may be combined into the 'supercoordinates'

$$X^a(s,\theta,\theta^*)=x^a(s)+\theta\xi^{*a}(s)+\xi^a(s)\theta^*+y^a(s)\theta\theta^*. \tag{2.1}$$

The pseudoclassical Lagrangian for a free superparticle is

$$L_0=\tfrac{1}{2}\int \mathrm{d}\theta\,\mathrm{d}\theta^*\ \eta_{ab}DX^aD^*X^b \tag{2.2}$$

$$=\tfrac{1}{2}\eta_{ab}[\dot{x}^a\dot{x}^b-\mathrm{i}(\xi^{*a}\dot{\xi}^b-\dot{\xi}^{*a}\xi^b)+y^ay^b] \tag{2.3}$$

where

$$D=\frac{\partial}{\partial\theta}+\mathrm{i}\theta^*\frac{\partial}{\partial s} \tag{2.4}$$

$$D^*=-\frac{\partial}{\partial\theta^*}-\mathrm{i}\theta\frac{\partial}{\partial s} \tag{2.5}$$

are the supercovariant derivatives with respect to the superspace transformations $s\to s+\mathrm{i}(\theta\varepsilon^*-\varepsilon\theta^*)$, $\theta\to\theta+\varepsilon$, $\theta^*\to\theta^*+\varepsilon^*$ that correspond to the off-shell supersymmetry of (2.3). In the graded algebra defined by the Poisson bracket

$$[A,B\}=\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a}-\frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a}+\mathrm{i}\eta^{ab}A\left(\frac{\overleftarrow{\partial}}{\partial\xi^a}\frac{\overrightarrow{\partial}}{\partial\xi^{*b}}+\frac{\overleftarrow{\partial}}{\partial\xi^a}\frac{\overrightarrow{\partial}}{\partial\xi^{*b}}\right)B \tag{2.6}$$

(which can be derived as a generalised Dirac bracket (Dirac 1958, Casalbuoni 1976) just as in the $N=1$ case (Rumpf 1982)) Lorentz transformations are generated by

$$J^{ab}=L^{ab}+S^{ab} \tag{2.7}$$

$$L^{ab}=x^ap^b-x^bp^a \tag{2.8}$$

$$S^{ab}=\mathrm{i}(\xi^a\xi^{*b}-\xi^b\xi^*a). \tag{2.9}$$

Note that the orbital angular momentum L^{ab} and the spin angular momentum S^{ab} are separately conserved, but that only their sum J^{ab} is invariant under supersymmetry transformations. For later use we introduce the generators T^{ab} of dilations and 'shears' of the spin variables

$$T^{ab} = \xi^a \xi^{*b} + \xi^b \xi^{*a} \tag{2.10}$$

$$[\xi^a, T^{bc}\} = -\mathrm{i}(\eta^{ac}\xi^b + \eta^{ab}\xi^c). \tag{2.11}$$

T^{ab} will be interpreted as a quadrupole moment tensor in §§ 4 and 5.

In canonical quantisation the Poisson bracket is replaced by the graded commutator multiplied by $\mathrm{i}/\hbar$. (For a discussion of quantisation directly in superspace we refer the reader to de Azcárraga *et al* (1986).) Consequently the quantum spin variables $\hat{\xi}^a$ and $(\xi^{*a})\hat{} \equiv \hat{\xi}^{\dagger a}$ obey the Clifford algebra relations

$$\{\hat{\xi}^a, \hat{\xi}^{\dagger b}\} = \hbar\eta^{ab}. \tag{2.12}$$

Therefore the representation of the spin degrees of freedom at the quantum level is sixteen dimensional. This representation is best derived in the form of a 'pseudo-Schrödinger' representation on the space of analytic functions of the ξ variables. Any such function is of the form

$$f(\xi) = A^{(0)} + \xi^i A_i^{(1)} + \frac{1}{2!}\xi^i\xi^j A_{ij}^{(2)} + \frac{1}{3!}\xi^i\xi^j\xi^k A_{ijk}^{(3)} + \frac{1}{4!}\xi^i\xi^j\xi^k\xi^l A_{ijkl}^{(4)} \tag{2.13}$$

where the $A^{(p)}$ take values in a Grassmann algebra and transform as antisymmetric tensors under the Lorentz group (such that f is invariant). Only later shall we restrict ourselves to complex-valued $A^{(p)}$. The natural operator realisation of $\hat{\xi}$ and $\hat{\xi}^\dagger$ is given by

$$\hat{\xi}^a f(\xi) = \hbar^{1/2}\xi^a f(\xi) \tag{2.14}$$

$$\hat{\xi}^{a\dagger} f(\xi) = \hbar^{1/2}\eta^{ab}\frac{\partial}{\partial\xi^b} f(\xi). \tag{2.15}$$

The scalar product in this representation space is determined (up to a constant factor) by the requirement that $\hat{\xi}^\dagger$ be the adjoint of $\hat{\xi}$:

$$f_1^* \cdot f_2 = -\int \prod_a (\mathrm{d}\xi^a\, \mathrm{d}\xi^{*a}) \exp(\xi^{*b}\xi_b) f_1^* f_2. \tag{2.16}$$

The representation just constructed is naturally extended to one comprising also the translational degrees of freedom by letting the A become tensor fields on spacetime. Then the 4-momentum operator is given by

$$\hat{p}_a = \mathrm{i}\hbar\partial_a \tag{2.17}$$

and is self-adjoint with respect to the scalar product

$$\langle f_1 | f_2 \rangle = \int \mathrm{d}^4x\, f_1^*(x) \cdot f_2(x). \tag{2.18}$$

The 'wavefunction' f may be equivalently represented by the direct sum of the tensor fields $A^{(p)}$. In the following we shall employ for conciseness the calculus of differential forms, in which this direct sum is denoted as an inhomogeneous differential form:

$$\psi = A^{(0)} + A_i^{(1)}\,\mathrm{d}x^i + \frac{1}{2!}A_{ij}^{(2)}\,\mathrm{d}x^i \wedge \mathrm{d}x^j + \frac{1}{3!}A_{ijk}^{(3)}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$
$$+ \frac{1}{4!}A_{ijkl}^{(4)}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l. \tag{2.19}$$

We now restrict ourselves to complex-valued fields $A^{(p)}$ and thus identify the quantum mechanical representation space of our system with the exterior algebra $E(M)$ of complex differential forms on Minkowski space. The scalar product of (2.18) and (2.16) implies the standard scalar product in $E(M)$ (in the following we suppress the superscript (p) when this is possible without ambiguities):

$$\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^4x \sum_{p=0}^{4} \frac{1}{p!} A^*_{i_1\ldots i_p}(x) A^{i_1\ldots i_p}(x) \equiv \int \mathrm{d}^4x\, \psi_1^* \cdot \psi_2. \tag{2.20}$$

Equations (2.14) and (2.15) imply that the action of $\hat{\xi}^a$ and $\hat{\xi}^{a\dagger}$ on ψ is

$$\hat{\xi}^a\psi = \hbar^{1/2}\, \mathrm{d}x^a \wedge \psi \tag{2.21}$$

$$\hat{\xi}^{a\dagger}\psi = \hbar^{1/2}\, \mathrm{d}x^a \lrcorner\, \psi \tag{2.22}$$

i.e. the exterior and interior product, respectively, with $\mathrm{d}x^a$. It can be checked that the operators S^{ab} corresponding to (2.9) indeed generate Lorentz transformations of ψ.

The supercharges $\hat{Q}$ and $\hat{Q}^\dagger$ corresponding to the pseudoclassical supersymmetry generators for the Lagrangian (2.2) are proportional to the exterior derivative d and the co-derivative δ, respectively:

$$\hat{Q} = \hat{p}_a\hat{\xi}^a = \mathrm{i}\hbar^{3/2}\mathrm{d} \tag{2.23}$$

$$\hat{Q}^\dagger = \hat{p}_a\hat{\xi}^{\dagger a} = \mathrm{i}\hbar^{3/2}\delta. \tag{2.24}$$

For the component fields this implies

$$\hat{Q}:\quad (A, A_i, A_{ij}, A_{ijk}, A_{ijkl}) \to \mathrm{i}\hbar^{3/2}(0, A_{,i}, 2A_{[j,i]}, 3A_{[jk,i]}, 4A_{[jkl,i]}) \tag{2.25}$$

$$Q^\dagger:\quad (A, A_i, A_{ij}, A_{ijk}, A_{ijkl}) \to \mathrm{i}\hbar^{3/2}(A^b{}_{,b}, A^b{}_{i,b}, A^b{}_{ij,b}, A^b{}_{ijk,b}, 0). \tag{2.26}$$

The last operator we have to identify is the free Hamiltonian $\hat{H} = \hat{p}^2/2$ generating the evolution in the parameter s. It is obviously realised by

$$\hat{H} = -(\hbar^2/2)\Box. \tag{2.27}$$

Note that the well known formula

$$\Box = \mathrm{d}\delta + \delta\mathrm{d} \tag{2.28}$$

corresponds exactly to the basic $N=2$ supersymmetry relation

$$2\hbar\hat{H} = \hat{Q}\hat{Q}^\dagger + \hat{Q}^\dagger\hat{Q}. \tag{2.29}$$

Experimental evidence suggests that physical states obey a mass-shell condition

$$\hat{H}\psi = -(m^2/2)\psi \qquad m^2 \geqslant 0. \tag{2.30}$$

However, this condition is not sufficient for the identification of physical states since we note that the scalar product (2.20) is indefinite even with the condition (2.30) (and even if restricted to positive frequency wavefunctions). This defect can be remedied in two possible ways. The first possibility is to impose the Lorentz condition

$$\delta\psi = 0 \tag{2.31}$$

on the wavefunctions. It is then easy to show using (2.26) and the Fourier transform that the scalar product (2.20) restricted to the subspace defined by (2.31) is positive definite for $m^2 > 0$ and positive semidefinite for $m^2 = 0$, provided we choose the metric signature $(-+++)$. (With the opposite signature convention, the scalar product (2.20) has to be modified by multiplying the summands with $(-1)^p$. This corresponds to a slightly different operator representation of the Grassmann variables, e.g. $\hat{\xi}^a = -\mathrm{i}\hbar^{1/2}\,\mathrm{d}x^a\wedge$, $\hat{\xi}^{a\dagger} = -\mathrm{i}\hbar^{1/2}\,\mathrm{d}x^a \llcorner$, $\hat{Q} = \hbar^{3/2}\,\mathrm{d}$, $\hat{Q}^\dagger = \hbar^{3/2}\delta$.) In the massless case positive definiteness can be achieved by forming the quotient with respect to the subspace defined by the equations

$$\psi = \mathrm{d}\lambda \qquad \delta\,\mathrm{d}\lambda = 0. \tag{2.32}$$

Of course the operation $\psi \to \psi + \mathrm{d}\lambda$ on massless antisymmetric tensor fields is just a gauge transformation according to the usual terminology, and forming the quotient with respect to (2.32) corresponds to removing the remaining gauge freedom after the Lorentz gauge condition has been imposed. In the supersymmetric formalism the same state of affairs is described as follows. Massive physical states are partially supersymmetric, i.e. they obey

$$\hat{Q}^\dagger\psi = 0. \tag{2.31$'$}$$

The action of the second supersymmetry generator $\hat{Q}$ leads out of the physical state space. In the massless case, however, this action amounts just to a gauge transformation, and therefore the gauge-invariant states obtained by forming equivalence classes are fully supersymmetric.

The other possible definition of physical states is related to the one just discussed by Hodge duality and amounts to exchanging the roles of $\hat{Q}$ and $\hat{Q}^\dagger$. As the only difference between the two representations is the parity of their states, we shall in the following stick to the quantum subsidiary condition (2.31).

If $m^2 \neq 0$, equations (2.30) and (2.31) follow from a variational principle based on the action

$$S[\psi] = \tfrac{1}{2}\langle\psi|\delta\mathrm{d} - m^2/\hbar^2|\psi\rangle \tag{2.33}$$

$$= -\tfrac{1}{2}\langle\mathrm{d}\psi|\mathrm{d}\psi\rangle - (m^2/2\hbar^2)\langle\psi|\psi\rangle + \text{surface terms}. \tag{2.34}$$

It has been shown by Capri and Kobayashi (1985a) that (2.34) and its 'dual' version (the kinetic operator $\delta\mathrm{d}$ replaced by $\mathrm{d}\delta$) are the only possible actions for antisymmetric tensor fields with mass. Moreover the operators in the corresponding Lagrange densities are, up to a factor $1/m^2$, each other's Klein–Gordon divisors (Capri and Kobayashi 1985b):

$$(\delta\mathrm{d} - m^2)(\mathrm{d}\delta - m^2) = -m^2(\Box - m^2). \tag{2.35}$$

If $m^2 = 0$, the action (2.34) is gauge invariant and implies only the Maxwell-like equation

$$\delta\mathrm{d}\psi = 0 \tag{2.36}$$

which has to be supplemented by the gauge condition (2.31) for consistency of the first quantisation.

It is straightforward to determine the particle content of the theory defined by (2.34) for $m^2 \neq 0$, since in this case the little group of the Poincaré group is O(3) and it suffices to count the number of independent components of ψ. The result is two spin-0 particles ($A^{(0)}$ and $A^{(3)}$) and two spin-1 particles ($A^{(1)}$ and $A^{(2)}$, the former being the well known Proca field). $A^{(4)}$ has no dynamics at all. In the massless case one counts the number of independent components of the field strength $\mathrm{d}\psi$ (Sezgin and van Nieuwenhuizen 1980, Tokuoka 1982) and finds that $A^{(0)}$ and $A^{(2)}$ describe helicity-0 particles, while $A^{(3)}$ does not propagate and $A^{(4)}$ drops out again. $A^{(1)}$ is, of course, the Maxwell field of helicity 1. The transition from $m^2=0$ to $m^2>0$ is thus accompanied by the well known phenomenon of 'spin-jumping' (Deser *et al* 1981).

For the sake of completeness we add some remarks on the relationship between the present formalism and the $N=1$ supersymmetry case. First of all, it is possible to obtain the antisymmetric tensor field equations for $m^2>0$ from the Dirac-like equation

$$\hat{Q}_\alpha \phi = \pm \mathrm{i}\frac{m}{2\hbar^{1/2}}\phi \tag{2.37}$$

where α is either 1 or 2 and $\hat{Q}_1$ and $\hat{Q}_2$ are the Hermitian and antiHermitian part, respectively, of $\hat{Q}$. If we set

$$\phi = A^{(p)} + B^{(p+1)} \tag{2.38}$$

then (2.37) implies (for $\alpha=1$)

$$\delta A^{(p)} = 0 \tag{2.39}$$

$$\mathrm{d}B^{(p+1)} = 0 \tag{2.40}$$

$$\mathrm{d}A^{(p)} = \pm m B^{(p+1)} \tag{2.41}$$

$$\delta B^{(p+1)} = \pm m A^{(p)}. \tag{2.42}$$

The last two equations imply

$$\delta\, \mathrm{d}A^{(p)} = -m^2 A^{(p)} \tag{2.43}$$

which together with (2.39) is the desired result. Note that the scalar product (2.20) is not positive on the forms ϕ obeying (2.42) (they are related on-shell to the wavefunctions ψ considered above by $\phi = \psi + (\mathrm{i}/m)\,\mathrm{d}\psi$). There exists however a different scalar product that is positive on positive frequency wavefunctions. This is the scalar product associated with the Kähler–Dirac interpretation (Benn and Tucker 1983) of (2.37). In this interpretation $E(M)$ is split into the direct sum of four representations of spin-$\frac{1}{2}$ (rather than the spin-0$\oplus$spin-1$\oplus$spin-1$\oplus$spin-0 of above) and $\frac{1}{2}(\hat{\xi}^a + \hat{\xi}^{a\dagger})$ act as the Dirac matrices γ^a. Since the ξ^a and ξ^{*a} cannot be realised individually in the Kähler–Dirac representation, it carries only $N=1$ supersymmetry.

We close with a remark concerning the 'bosonic' and 'fermionic' states with respect to the supersymmetry. In the one-dimensional case these are defined by the spaces $\ker \hat{Q} = \mathrm{i}m\hat{Q}$ and $\ker \hat{Q}^\dagger = \mathrm{i}m\hat{Q}^\dagger$, respectively. In higher dimensions this definition is not viable, since $E(M) = \mathrm{i}m\hat{Q} \oplus \mathrm{i}m\hat{Q}^\dagger$ only in the case of trivial cohomology. Moreover cohomology is not even defined in the case of interest here, since Hodge theory requires a Hilbert space structure of $E(M)$ and has thus been confined to compact Riemannian manifolds. Thus the only natural notion of fermion number is the rank p of a homogeneous differential form and the fermion number operator is given by

$$\hat{F} = \eta_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b \lrcorner . \tag{2.44}$$

In particular, even forms are 'bosonic' and odd forms 'fermionic'. Note that for the free theory the number of spin degrees of freedom determined above is exactly equal in the bosonic and the fermionic sector. Thus the formal analogue of the Witten index $(-1)^F$ (which can be defined rigorously only in the compact Riemannian case, where it yields the Euler number (Alvarez-Gaumé 1983)) vanishes in this case.

3. Supersymmetric coupling to a scalar field

There are three types of external fields—scalar, vector and tensor—to which the system described by the free Lagrangian (2.3) may be coupled in a supersymmetric manner. We first consider the scalar case. The manifestly supersymmetric Lagrangian is

$$L_W = \int \mathrm{d}\theta\, \mathrm{d}\theta^*[\tfrac{1}{2}DX_a D^* X^a + W(X)] \tag{3.1}$$

$$= \tfrac{1}{2}[\dot{x}^2 - \mathrm{i}(\xi^a \dot{\xi}^*_a - \dot{\xi}^a \xi^*_a) + y^2 - 2W_{,a}y^a + W_{,ab}T^{ab}] \tag{3.2}$$

implying the equations of motion

$$y_a = W_{,a} \tag{3.3}$$

$$\ddot{x}_a = -\tfrac{1}{2}(W_{,c}W^{,c})_{,a} + \tfrac{1}{2}W_{,bca}T^{bc} \tag{3.4}$$

$$\dot{\xi}_a = \mathrm{i}\,W_{,ab}\xi^b. \tag{3.5}$$

The last equation implies

$$\dot{S}_{ab} = W_{,cb}T_a^{\ c} - W_{,ca}T_b^{\ c} \tag{3.6}$$

$$\dot{T}_{ab} = W_{,ac}S^c_{\ b} + W_{,bc}S^c_{\ a}. \tag{3.7}$$

Eliminating the y variables from the Lagrangian (3.2) via (3.3) we may construct the Hamiltonian and the supercharges

$$H_W = \tfrac{1}{2}p^2 + \tfrac{1}{2}W_{,c}W^{,c} - \tfrac{1}{2}W_{,ab}T^{ab} \tag{3.8}$$

$$Q_W = (p_a - \mathrm{i}\,W_{,a})\xi^a. \tag{3.9}$$

We now turn to the quantisation of the system. First we note that

$$\hat{H}_W = \tfrac{1}{2}\hbar^{-1}(\hat{Q}_W\hat{Q}^\dagger_W + \hat{Q}^\dagger_W\hat{Q}_W) = -\tfrac{1}{2}[\hbar^2\Box - W_{,c}W^{,c} - \hbar(\Box W)] - \hbar W_{,ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b \lrcorner. \tag{3.10}$$

Comparison with (3.8) shows that the natural factor ordering of the quadrupole moment tensor is

$$\hat{T}^{ab} = \tfrac{1}{2}([\hat{\xi}^a, \hat{\xi}^{\dagger b}] + [\hat{\xi}^b, \hat{\xi}^{\dagger a}]) = \hbar(\tfrac{1}{2}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \lrcorner + \tfrac{1}{2}\,\mathrm{d}x^b \wedge \mathrm{d}x^a \lrcorner - \eta^{ab}). \tag{3.11}$$

It is natural to consider

$$\hat{Q}^\dagger_W \psi \equiv \mathrm{i}\hbar^{1/2}(\hbar\delta - \mathrm{d}W\lrcorner)\psi = 0 \tag{3.12}$$

as the candidate for a consistent quantisation condition. In order to examine the positivity of the scalar product (2.20) we recall that in the case of vanishing external field an equivalent scalar product for positive freqency solutions with $m^2 \geq 0$ is provided by the charge form

$$(\psi_1, \psi_2) = \int_\Sigma \mathrm{d}\sigma_a\, j^a(\psi_1, \psi_2) \tag{3.13}$$

where Σ is a spacelike Cauchy hypersurface and

$$j_a(\psi_1, \psi_2) = \mathrm{i}\psi_1^* \cdot \overleftrightarrow{\partial}_a \psi_2 \tag{3.14}$$

is the conserved current implied by the mass-shell condition (2.30). Equation (3.13) is the scalar product usually considered in second quantisation. The exact relationship between (2.20) and (3.13) in the free field case is

$$\pm\langle {}^\pm\psi_{m^2}, {}^\pm\psi_{m'^2}\rangle = 2\pi\delta(m^2 - m'^2)({}^\pm\psi_{m^2}, {}^\pm\psi_{m^2}) \tag{3.15}$$

where ${}^{+}\psi_{m^2}$, ${}^{-}\psi_{m^2}$ are a positive and negative frequency solution, respectively, of the mass-shell condition with $m^2 \geqslant 0$. Note that

$$(\psi_1^*, \psi_2^*) = -(\psi_1, \psi_2) \tag{3.16}$$

and hence (,) is negative for negative frequency solutions.

In the presence of an external scalar field the charge form is still given by (3.13) and (3.14). However, its relationship to the scalar product (2.20) is less straightforward. (This fact reflects the well known problem of the identification of a physical Fock representation in second quantisation (Rumpf and Urbantke 1978).) We shall therefore confine ourselves in the following to the case that W becomes asymptotically constant in one time direction, say, in the future. Let us denote now by ${}^{+}\psi_{m^2}$ those solutions of the mass-shell condition that contain only positive frequency contributions asymptotically. Then $({}^{+}\psi_{m^2}, {}^{+}\psi_{m^2}) > 0$ if ${}^{+}\psi_{m^2} \neq 0$ and moreover ${}^{+}\psi_{m^2}$ may be chosen such that its charge is finite. Because of the positive frequency character of ${}^{+}\psi_{m^2}$ it may be embedded into a z-analytic family $\{{}^{+}\psi_z\}_{z\in C}$ of solutions of

$$(\hat{H}_W + z/2)\psi_z = 0 \qquad \hat{Q}^{\dagger}_W \psi_z = 0 \tag{3.17}$$

such that for $\operatorname{Im} z > 0$ ${}^{+}\psi_z$ vanishes exponentially for $t \to \infty$ and its spatial integrability is unchanged. Hence

$$({}^{+}\psi_{m^2-\mathrm{i}\varepsilon}, {}^{+}\psi_{m^2-\mathrm{i}\varepsilon}) \xrightarrow{t\to\infty} 0 \qquad \varepsilon > 0. \tag{3.18}$$

Of course for complex z the charge form (3.13) is no longer independent of Σ. Specifically

$$\psi^*_{m^2-\mathrm{i}\varepsilon} \cdot \hat{H}_W \psi_{m^2-\mathrm{i}\varepsilon} - (\hat{H}_W \psi_{m^2-\mathrm{i}\varepsilon})^* \cdot \psi_{m^2-\mathrm{i}\varepsilon} = -2\mathrm{i}\varepsilon \psi^*_{m^2-\mathrm{i}\varepsilon} \cdot \psi_{m^2-\mathrm{i}\varepsilon} \tag{3.19}$$

implies

$$\partial_a j^a(\psi_{m^2-\mathrm{i}\varepsilon}, \psi_{m^2-\mathrm{i}\varepsilon}) = -2\hbar^2 \varepsilon \psi^*_{m^2-\mathrm{i}\varepsilon} \cdot \psi_{m^2-\mathrm{i}\varepsilon}. \tag{3.20}$$

Therefore

$$\int_\Sigma^\infty \mathrm{d}t\, \mathrm{d}^3x\, \psi^*_{m^2-\mathrm{i}\varepsilon} \cdot \psi_{m^2-\mathrm{i}\varepsilon} = \frac{1}{2\hbar^2\varepsilon} (\psi_{m^2-\mathrm{i}\varepsilon}, \psi_{m^2-\mathrm{i}\varepsilon})_\Sigma > 0. \tag{3.21}$$

The inequality holds for sufficiently small ε. Taking the limit $\varepsilon \to 0$ we obtain that for any spacelike hypersurface Σ

$$\int_\Sigma^\infty \mathrm{d}t\, \mathrm{d}x\, \psi^*_{m^2} \cdot \psi_{m^2} \geqslant 0. \tag{3.22}$$

Thus the scalar product obeys

$$\langle \psi_{m^2} | \psi_{m^2} \rangle \geqslant 0 \tag{3.23}$$

(it will in fact diverge). A similar argument for asymptotic negative-frequency solutions ${}^{-}\psi_{m^2}$ (which may be chosen as ${}^{+}\psi^*_{m^2}$) establishes that $\langle\ ,\ \rangle$ is positive on both ${}^{+}\psi_{m^2}$ and ${}^{-}\psi_{m^2}$ for $m^2 \geqslant 0$. The latter generate, by definition, the space of physical states. From the proof it can be seen that it is even possible to relax the asymptotic condition on the external field. All that is really required is the existence of solutions ${}^{\pm}\psi_{m^2}$ (or alternatively ${}_{\pm}\psi_{m^2}$) that have the appropriate sign of charge and possess analytic extensions ${}^{\pm}\psi_{m^2\mp\mathrm{i}\varepsilon}({}_{\pm}\psi_{m^2\pm\mathrm{i}\varepsilon})$ fulfilling the necessary fall-off conditions in the future (past). This general criterion for the consistency of the partial supersymmetry condition (3.12) will be of special interest in the case of an external gravitational field.

Supersymmetry also implies in the presence of an external field that both the mass-shell condition and the subsidiary condition can be derived from a variational principle if $m^2 > 0$. For the scalar field coupling the action is

$$S_W[\psi] = -\frac{1}{2\hbar^2}\langle\psi|\frac{1}{\hbar}\hat{Q}^\dagger_W\hat{Q}_W + m^2|\psi\rangle$$
$$= -\tfrac{1}{2}\langle(\mathrm{d}+\hbar^{-1}\,\mathrm{d}W\wedge)\psi|(\mathrm{d}+\hbar^{-1}\,\mathrm{d}W\wedge)\psi\rangle - \frac{m^2}{2\hbar^2}\langle\psi|\psi\rangle. \qquad (3.24)$$

In which sense are the pseudoclassical equations of motion (3.3)-(3.5) the classical limit of the quantum theory just described? We observe that the Heisenberg operator equations of motion implied by the quantum Hamiltonian (3.10) are formally identical with the pseudoclassical equations of motion. However, we refrain from considering them as a viable classical limit, as we gather from experience that only even observables are actually measurable and that the results of measurements are always described in terms of real numbers. We propose therefore that the classical limit equations should involve only even observables and be obtained by replacing quantum operators by their expectation values and expectation values of products by the product of the expectation values of their even factors (note that the expectation values of odd observables vanish for states of definite F). In this sense equations (3.4), (3.6) and (3.7) constitute indeed the classical particle limit of the dynamics of antisymmetric tensor fields in an external scalar field. A physical intuition for the observable T_{ab} will be developed only in the subsequent sections.

Finally we note that the standard scalar potential in scalar field theory is related to the superpotential W by

$$V = W_{,c}W^{,c} - \hbar(\Box W). \qquad (3.25)$$

For given V the solutions W of (3.25) will not necessarily be real. But the reality of W is necessary for the self-adjointness of $\hat{H}_W$ and the definition of the charge form (3.13). Thus not every scalar-coupled scalar field can be interpreted in the supersymmetric way.

4. External gravitation

It is possible to couple the point particle in an $N = 2$ supersymmetric way to a complex Hermitian tensor field of rank 2. In the present section we confine ourselves to the case that the tensor field is real and symmetric. It may thus be identified with the pseudoRiemannian metric tensor field $g_{ab}(x)$ of general relativity. The corresponding manifestly supersymmetric particle Lagrangian is

$$L_g = \tfrac{1}{2}\int \mathrm{d}\theta\,\mathrm{d}\theta^*\, g_{ab}(X)DX^aD^*X^b$$
$$= \tfrac{1}{2}\{g_{ab}(x)[\dot{x}^a\dot{x}^b - \mathrm{i}(\xi^a\dot{\xi}^{*b} - \dot{\xi}^a\xi^{*b}) + y^ay^b] - g_{ab,c}\dot{x}^aS^{bc}$$
$$-\{cab\}y^cT^{ab} - \tfrac{1}{4}g_{ab,cd}T^{ab}T^{cd}\}. \qquad (4.1)$$

Here $\{cab\}$ denotes the Christoffel symbol of the first kind

$$\{cab\} = \tfrac{1}{2}(g_{ca,b} + g_{cb,a} - g_{ab,c}). \qquad (4.2)$$

The Lagrangian L_g is formally identical with the Lagrangian of the one-dimensional version of the $N=2$ (sometimes also called $N=1$) supersymmetric non-linear σ model that has been studied intensively in the literature (Davis *et al* 1984). Our point of view is, however, that (4.1) describes physical particles in curved spacetime. It will be shown that upon first quantisation L_g yields the minimally coupled antisymmetric tensor field equations.

As in the scalar field case the equation of motion for y

$$y^a = \frac{1}{2}\left\{ \begin{matrix} a \\ b \quad c \end{matrix} \right\} T^{bc} \tag{4.3}$$

may be used to simplify L_g. The result is

$$L_g = \tfrac{1}{2} g_{ab}\left[\dot{x}^a \dot{x}^b - \mathrm{i}\left(\xi^a \frac{\mathrm{D}\xi^{*b}}{\mathrm{d}s} - \frac{\mathrm{D}\xi^a}{\mathrm{d}s}\xi^{*b}\right)\right] - \tfrac{1}{4} R_{abcd}\xi^a \xi^b \xi^{*c} \xi^{*d} \tag{4.4}$$

which is manifestly scalar with $R^a{}_{bcd}$ the Riemann–Christoffel tensor. The remaining equations of motion are

$$\ddot{x}^a + \left\{ \begin{matrix} a \\ b \quad c \end{matrix} \right\} \dot{x}^b \dot{x}^c = -\tfrac{1}{2} R^a{}_{bcd} \dot{x}^b S^{cd} + \tfrac{1}{4} R_{ijkl}{}^{;a} \xi^i \xi^j \xi^{*k} \xi^{*l} \tag{4.5}$$

$$\frac{\mathrm{D}\xi^a}{\mathrm{d}s} = -\mathrm{i} R_{ijk}{}^a \xi^i \xi^j \xi^{*k}. \tag{4.6}$$

Thus the supersymmetric particle has the characteristics of a mass pole, dipole and quadrupole.

The Lagrangian (4.1) may be subjected to the canonical formalism. The appearance of the general spacetime metric g_{ab} requires a change in the definition of the generalised Poisson bracket (2.6), as for consistency

$$[\xi^a, \xi^{*b}\} = \mathrm{i} g^{ab}(x). \tag{4.7}$$

The modification of the bracket is similar to that encountered in the case of simple supersymmetric mechanics in curved spacetime. The reader is referred to Rumpf (1982) for the details. The Dirac formalism requires the introduction of an orthonormal tetrad field $e^\alpha{}_a(x)$ obeying

$$\eta_{\alpha\beta} e^\alpha{}_a e^\beta{}_b = g_{ab} \tag{4.8}$$

and the redefinition of the Poisson bracket in terms of the anholonomic Grassmann variables

$$\xi^\alpha = e^\alpha{}_a(x)\xi^a \tag{4.9}$$

such that

$$[\xi^\alpha, \xi^{*\beta}\} = \mathrm{i}\eta^{\alpha\beta} \tag{4.10}$$

$$[p_a, \xi^\alpha\} = 0. \tag{4.11}$$

The Hamiltonian corresponding to the Lagrangian (4.4) is

$$H_g = \tfrac{1}{2} g_{ab} \dot{x}^a \dot{x}^b - \tfrac{1}{4} R_{abcd} \xi^a \xi^{*c} \xi^b \xi^{*d} \tag{4.12}$$

where

$$\dot{x}^a = g^{ab}\left\{ p_b + \mathrm{i}\eta_{\alpha\gamma}\left\{ \begin{matrix} \gamma \\ \beta \quad b \end{matrix} \right\} \xi^\alpha \xi^{*\beta} \right\}. \tag{4.13}$$

Note the appearance of the anholonomic components of the Levi-Civita connection (also called Ricci rotation coefficients).

The supercharge Q is

$$Q=\left(p_a+\mathrm{i}\eta_{\alpha\gamma}\begin{Bmatrix}\gamma\\ \beta\ \ a\end{Bmatrix}\xi^\alpha\xi^{*\beta}\right)\xi^a. \tag{4.14}$$

In constructing a quantum representation of the pseudoclassical observables by operators one has to observe that these operators act on tensor densities of weight $\frac{1}{4}$ rather than on tensor fields. Only in such a representation is the canonical momentum operator (2.17) self-adjoint. This fact is, in principle, well known from relativistic quantum mechanics in curved spacetime (Rumpf 1982). In the present case it is moreover convenient to refer the tensor indices of the wavefunctions to the orthonormal tetrad field $e^\alpha{}_a$ introduced in (4.8) in order to repeat the reasoning employed in § 2 for the construction of the state space. We shall thus introduce the wavefunctions

$$\tilde{\psi}=|g|^{1/4}\psi \tag{4.15}$$

and use components

$$\tilde{A}_{\alpha_1\dots\alpha_p}=|g|^{1/4}e_{\alpha_1}{}^{i_1}\dots e_{\alpha_p}{}^{i_p}A_{i_1}\dots{}_{i_p} \tag{4.16}$$

where

$$g\equiv\det(g_{ab})=-(\det e^\alpha{}_\beta)^2\equiv -e^2 \tag{4.17}$$

$$e_\alpha{}^c e^\beta{}_c=\delta_\alpha{}^\beta. \tag{4.18}$$

The correct scalar product is then obtained from (2.20) by replacing $A^{(p)}$ by $\tilde{A}^{(p)}$. Obviously we have

$$\hat{\xi}^\alpha=e^\alpha\wedge\equiv e^\alpha{}_a(x)\,\mathrm{d}x^a\wedge \tag{4.19}$$

$$\hat{\xi}^{\dagger\alpha}=e^\alpha\lrcorner. \tag{4.20}$$

Note that

$$\hat{p}_a(\tilde{A}_{\alpha_1\dots\alpha_p}e^{\alpha_1}\wedge\dots\wedge e^{\alpha_p})=\mathrm{i}\hbar\tilde{A}_{\alpha_1\dots\alpha_p,a}e^{\alpha_1}\wedge\dots\wedge e^{\alpha_p}. \tag{4.21}$$

For the proper definition of the quantum supercharges we observe that there is no factor ordering ambiguity in

$$\hat{x}_a=\hat{p}_a+\mathrm{i}\eta_{\alpha\gamma}\begin{Bmatrix}\gamma\\ \beta\ \ a\end{Bmatrix}\hat{\xi}^\alpha\hat{\xi}^{\dagger\beta} \tag{4.22}$$

since the connection { } is antisymmetric in the anholonomic indices and because of (2.12). On the other hand, it is well known from the scalar particle theory in curved spacetime that the correct factor ordering of the scalar part of the mass operator is

$$\hat{H}_g^{(\mathrm{sc})}=\tfrac{1}{2}|g|^{-1/4}\hat{p}_a|g|^{1/2}g^{ab}\hat{p}_b|g|^{-1/4}. \tag{4.23}$$

This fixes the factor ordering of the supercharge $\hat{Q}_g$ as

$$\hat{Q}_g=\hat{\xi}^a|g|^{1/4}\hat{x}_a|g|^{-1/4}. \tag{4.24}$$

Now $|g|^{1/4}\hat{x}_a|g|^{-1/4}$ acts on $\tilde{\psi}$ as

$$\begin{aligned}|g|^{1/4}\hat{x}_a|g|^{-1/4}:\quad \tilde{A}_{\alpha_1\dots\alpha_p}\to\mathrm{i}\hbar|g|^{1/4}\Bigg(&A_{\alpha_1\dots\alpha_p,a}-\begin{Bmatrix}\beta\\ \alpha_1\ \ a\end{Bmatrix}A_{\beta\alpha_2\dots\alpha_p}\\ &-\dots-\begin{Bmatrix}\beta\\ \alpha_p\ \ a\end{Bmatrix}A_{\alpha_1\dots\alpha_{p-1}\beta}\Bigg)\\ =\mathrm{i}\hbar|g|^{1/4}\nabla_a A_{\alpha_1\dots\alpha_p}&\end{aligned} \tag{4.25}$$

i.e. as $\mathrm{i}\hbar$ times the covariant derivative on ψ. Therefore

$$\hat{Q}_g\tilde{\psi} = \mathrm{i}\hbar^{3/2}|g|^{1/4}\,\mathrm{d}\psi \tag{4.26}$$

where the connection drops out, because it is symmetric in the two lower indices when referred to the coordinate basis. Likewise one finds

$$\begin{aligned}\hat{Q}_g^\dagger\colon\quad \tilde{A}_{\alpha_1\ldots\alpha_{p-1}} \to \mathrm{i}\hbar^{3/2}|g|^{1/4}\bigg(&|g|^{-1/2}\partial_a(|g|^{1/2}A^a{}_{\alpha_2\ldots\alpha_p})\\ &-\begin{Bmatrix}\beta\\ \alpha_2\ a\end{Bmatrix}A^a{}_{\beta\alpha_3\ldots\alpha_p}-\ldots-\begin{Bmatrix}\beta\\ \alpha_p\ a\end{Bmatrix}A^a{}_{\alpha_2\ldots\alpha_{p-1}\beta}\bigg)\\ =\mathrm{i}\hbar^{3/2}|g|^{1/4}\nabla_aA^a{}_{\alpha_2\ldots\alpha_p}&\end{aligned} \tag{4.27}$$

that is,

$$\hat{Q}_g^\dagger\tilde{\psi} = \mathrm{i}\hbar^{3/2}|g|^{1/4}\delta_g\psi. \tag{4.28}$$

The quantum Hamiltonian implied by the supercharges is

$$\hat{H}_g = -(\hbar^2/2)|g|^{1/4}(\mathrm{d}\delta_g+\delta_g\,\mathrm{d})|g|^{-1/4} \tag{4.29}$$

$$= -(\hbar^2/2)|g|^{1/4}(\nabla_a\nabla^a + R_{abcd}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\lrcorner\,\mathrm{d}x^c\wedge\mathrm{d}x^d\lrcorner)|g|^{-1/4} \tag{4.30}$$

$$= \tfrac{1}{2}|g|^{-1/4}\dot{x}_a|g|^{1/2}g^{ab}\dot{x}_b|g|^{-1/4}+\tfrac{1}{8}R_{abcd}\hat{S}^{ab}\hat{S}^{cd}. \tag{4.31}$$

The two terms in (4.31) correspond exactly to those of (4.30). The correspondence of (4.31) with (4.12) becomes obvious upon using the identity

$$R_{abcd} = -R_{adbc} - R_{acdb} \tag{4.32}$$

obeyed by the Riemann tensor. Note that the Lichnerowicz Laplacian $\mathrm{d}\delta_g+\delta_g\mathrm{d}$ differs from the minimally coupled d'Alembertian $\nabla_a\nabla^a$ by the curvature term

$$R_{abcd}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\lrcorner\,\mathrm{d}x^c\wedge\mathrm{d}x^d\lrcorner = R_{ab}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\lrcorner - R_{abcd}\,\mathrm{d}x^a\wedge\mathrm{d}x^c\wedge\mathrm{d}x^b\lrcorner\,\mathrm{d}x^d\lrcorner \tag{4.33}$$

where

$$R_{ab} = R^c{}_{abc} \tag{4.34}$$

is the Ricci tensor.

Subjecting physical states to the subsidiary condition

$$\hat{Q}_g^\dagger\tilde{\psi} = 0 \tag{4.35}$$

one can show the positivity of the scalar product

$$\langle\psi_1|\psi_2\rangle_g = \langle\tilde{\psi}_1|\tilde{\psi}_2\rangle \tag{4.36}$$

in a non-trivial class of external gravitational fields by a straightforward generalisation of the argument used in § 4. We note that the charge form (,) is now defined via the current density

$$j_a(\psi_1,\psi_2) = \mathrm{i}\psi_1^*\cdot\overleftrightarrow{\nabla}_a\psi_2. \tag{4.37}$$

Both (4.35) and the mass-shell condition with $\hat{H}_g$ for $m^2>0$ follow from the action

$$S_g[\psi] = -\tfrac{1}{2}\langle\mathrm{d}\psi|\mathrm{d}\psi\rangle_g - \frac{m^2}{2\hbar^2}\langle\psi|\psi\rangle. \tag{4.38}$$

Regarding the Heisenberg equations of motion and the classical limit of the quantum theory just derived, similar remarks apply as made at the end of § 4. A new complication arises from the fact that the operators $\hat{\dot{x}}^a$ do not commute with the functions of $\hat{x}$ with which they are multiplied in the quantum version of the equation of motion (4.5). It is a simple but lengthy task to work out the exact factor ordering (Rumpf 1982). Modulo this factor ordering, however, the Heisenberg equations coincide formally with the pseudoclassical equations of motion. In terms of the even observables S_{ab} and T_{ab} they are

$$\mathrm{D}^2x^a/\mathrm{d}s^2 = -\tfrac{1}{2}R^a{}_{bcd}\dot{x}^bS^{cd} + \tfrac{1}{8}R_{ijkl}{}^{;a}S^{ij}S^{kl} \tag{4.39}$$

$$\mathrm{D}S^{ab}/\mathrm{d}s = R_{ijk}{}^{[a|}S^{ij}S^{k|b]} \tag{4.40}$$

$$\mathrm{D}T^{ab}/\mathrm{d}s = R_{ijk}{}^{(a|}S^{ij}T^{k|b)}. \tag{4.41}$$

5. Vector field coupling

In this section we consider the coupling of the supersymmetric particle to an external vector potential A_a. There is only one such coupling yielding a real Lagrangian, namely

$$L_A = \tfrac{1}{2}\int \mathrm{d}\theta\,\mathrm{d}\theta^*[\mathrm{D}X^a\mathrm{D}^*X_a + qA_a(X)\dot{X}^a] \tag{5.1}$$

$$= \tfrac{1}{2}[\dot{x}^2 - i(\xi^a\dot{\xi}^*_a - \dot{\xi}^a\xi^*_a) + y^2] + qA_a\dot{y}^a + qA_{a,b}\dot{x}^ay^b$$
$$- qA_{a,b}(\dot{\xi}^a\xi^{*b} + \xi^b\dot{\xi}^{*a}) - \tfrac{1}{2}qA_{a,bc}\dot{x}^aT^{bc}. \tag{5.2}$$

The equations of motion are

$$y^a = -qF^a{}_b\dot{x}^b \tag{5.3}$$

$$h_{ab}\ddot{x}^b + q^2(F_{ad}F^d{}_{b,c} + F_{ab,d}F^d{}_c)\dot{x}^b\dot{x}^c = -\tfrac{1}{2}qF_{ab,cd}\dot{x}^bT^{cd} + q^2F_{ba,k}\gamma^{bd}F_{id,j}S^{jk}\dot{x}^i \tag{5.4}$$

$$\dot{\xi}^a = \mathrm{i}\gamma^{ab}qF_{ib,j}\dot{x}^k\xi^i \tag{5.5}$$

where

$$F_{ab} = A_{b,a} - A_{a,b} \tag{5.6}$$

$$h_{ab} = \eta_{ab} + q^2F_{ac}F^c{}_b \tag{5.7}$$

$$\gamma_{ab} = \eta_{ab} + \mathrm{i}qF_{ab} \tag{5.8}$$

$$\gamma^{ac}\gamma_{cb} = \delta^a{}_b. \tag{5.9}$$

The appearance of the two 'metrics' h_{ab} and γ_{ab} is somewhat surprising at this point and will be traced back in the next section to the existence of a more geometrical version of the superspace action implied by (5.1). Note that because of (5.6) γ_{ab} is a complex Kähler metric. The two metrics are related by

$$h_{ab} = \gamma^*_{ac}\gamma_{db}\eta^{cd}. \tag{5.10}$$

A consequence of this equation that will be used later is

$$\gamma^{(ab)} = h^{ab}. \tag{5.11}$$

If A_a is tentatively identified with the electromagnetic potential, the weak coupling limit of equation (5.4),

$$\ddot{x}_a = -\tfrac{1}{2}qF_{ab,cd}T^{cd}\dot{x}^b \tag{5.12}$$

suggests the interpretation of the coupling constant q as the electric 'quadrupole strength' of the particle, qT_{ab} being the electric quadrupole moment tensor. The charge and the electric and magnetic dipole moments of the particle all vanish.

The equations of motion (5.4) and (5.5) can be interpreted in a purely geometrical manner. To see this we note that the same equations are obtained if (5.3) is substituted into the Lagrangian (5.2). Upon this substitution the Lagrangian becomes (up to a total derivative)

$$L_A = \tfrac{1}{2}[h_{ab}\dot{x}^a\dot{x}^b - \mathrm{i}\gamma_{ba}(\dot{\xi}^a\xi^{*b} - \xi^a\dot{\xi}^{*b}) + qF_{ab,c}\dot{x}^aT^{bc}]. \tag{5.13}$$

Let us write (5.13) in a manifestly covaraint way. To this end we have to introduce the Levi-Civita connection $\{_b{}^a{}_c\}$ of Minkowski space (which vanishes in cartesian coordinates) and to replace $\dot{\xi}^a$ by $\dot{\xi}^a + \{_b{}^a{}_c\}\xi^b\dot{x}^c$ and $F_{ab,c}$ by $F_{ab;c}$, where the semicolon denotes the covariant derivative with respect to the Levi-Civita connection. We obtain a more compact notation if we define also a complex connection $\Gamma^a{}_{bc}$ by

$$\Gamma^a{}_{bc} = \left\{ \begin{matrix} a \\ b \quad c \end{matrix} \right\} + \mathrm{i}q\gamma^{ad}F_{dc;b}. \tag{5.14}$$

This connection is neither torsion-free nor compatible with the Minkowski metric, but it is compatible with the metric γ_{ab} in the following sense:

$$\nabla_c\gamma_{\bar{a}b} = 0. \tag{5.15}$$

The bar over the first index of γ indicates that the complex conjugate Γ^* of Γ (which is again a connection) has to appear in the correction term with respect to this index in the covariant derivative

$$\nabla_c\gamma_{\bar{a}b} = \gamma_{ab,c} - \Gamma^{*d}{}_{ac}\gamma_{db} - \Gamma^d{}_{bc}\gamma_{ad}. \tag{5.16}$$

Equation (5.15) rests on the cyclic identity

$$F_{ab,c} + F_{ca,b} + F_{bc,a} = 0. \tag{5.17}$$

The torsion tensor

$$S^a{}_{bc} := \Gamma^a{}_{[bc]} \tag{5.18}$$

obeys the relation

$$\gamma_{ad}S^d{}_{bc} + \gamma_{da}S^{*d}{}_{bc} = 0. \tag{5.19}$$

The tensorial relations (5.15) and (5.19) do not determine $\Gamma^a{}_{bc}$, but only the real part of $\gamma_{ad}\Gamma^d{}_{bc}$.

Introducing the absolute derivative

$$\frac{\mathrm{D}\xi^a}{\mathrm{d}s} := \dot{\xi}^a + \Gamma^a{}_{bc}\xi^b\dot{x}^c \tag{5.20}$$

we can now write L_A in the form

$$L_A = \frac{1}{2}\left[h_{ab}\dot{x}^a\dot{x}^b - \mathrm{i}\gamma_{ba}\left(\xi^a\frac{\mathrm{D}\xi^{*b}}{\mathrm{d}s} - \frac{\mathrm{D}\xi^a}{\mathrm{d}s}\xi^{*b} \right) \right] \tag{5.21}$$

which is manifestly scalar with respect to general (real) coordinate transformations. This property of L_A implies that the equations of motion are tensorial. Indeed we may rewrite (5.4) and (5.5) in the manifestly covariant form

$$\ddot{x}^a + \{_b{}^a{}_c\}_h\dot{x}^b\dot{x}^c = \tfrac{1}{2}\mathrm{i}h^{al}(\gamma_{ci}R^{*c}{}_{jkl}\xi^i\xi^{*j} - \gamma^*_{ci}R^c{}_{jkl}\xi^j\xi^{*i})\dot{x}^k \tag{5.22}$$

$$\mathrm{D}\xi^a/\mathrm{d}s = 0. \tag{5.23}$$

Here $\{\,\}_h$ denotes the Christoffel symbol with respect to the metric h_{ab}, and $R^a{}_{bcd}$ is the curvature tensor of the connection Γ:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ic}\Gamma^i{}_{bd} - \Gamma^a{}_{id}\Gamma^i{}_{bc}. \tag{5.24}$$

We conclude from (5.22) and (5.23) that in the case of a constant electromagnetic field the particle follows a geodesic with respect to the metric h_{ab}, while its spin and 'quadrupole' tensors undergo parallel transport in the sense of Minkowski space geometry.

The Hamiltonian formalism based on the Lagrangian (5.13) necessitates a redefinition of the Poisson bracket (2.6) for similar reasons as observed in the preceding section. In order to have

$$[\xi^a, \xi^{*b}\} = \mathrm{i}\gamma^{ab}(x) \tag{5.25}$$

we introduce a complex orthonormal tetrad field $e_\alpha{}^a(x)$ obeying

$$e^{*a}_\alpha(x)\gamma_{ab}(x)e_\beta{}^b(x) = \rho_{\alpha\beta} \tag{5.26}$$

where $\rho_{\alpha\beta}$ is a constant diagonal quadratic form with entries $+1$ or -1 (for sufficiently small qF_{ab}, $\rho_{\alpha\beta} = \eta_{\alpha\beta}$). Up to now we have considered only real holonomic coordinate systems; they were referred to by roman indices. In the following we shall also use the complex anholonomic coordinate bases $\{e_\alpha\}$ and $\{e^*_\alpha\}$, and their duals. In order to avoid confusion we shall denote indices referring to e_α (or its dual) by Greek letters and indices referring to e^*_α (or its dual) by Greek letters carrying a bar. For instance

$$\gamma_{\bar\alpha\beta} = \rho_{\alpha\beta} \tag{5.27}$$

$$(\gamma^*)_{\alpha\beta} \neq (\gamma_{\alpha\beta})^* = (\gamma^*)_{\bar\alpha\bar\beta}. \tag{5.28}$$

In the following, the anholonomic Grassmann variables

$$\xi^\alpha = e^\alpha{}_a(x)\xi^a \tag{5.29}$$

$$\xi^{*\bar\alpha} = e^{*\alpha}{}_a(x)\xi^{*a} \tag{5.30}$$

will play a fundamental role. They have to be considered as canonical variables replacing the ξ^a and ξ^{*a}. They also appear in the correct bracket, which may be derived as the Dirac bracket of the constrained Hamiltonian dynamics defined by (5.13) (for a systematic derivation see Rumpf (1982)):

$$[A, B\} = \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} + \mathrm{i}\rho^{\alpha\beta}A\left(\frac{\overleftarrow{\partial}}{\partial\xi^\alpha}\frac{\overrightarrow{\partial}}{\partial\xi^{*\bar\beta}} + \frac{\overleftarrow{\partial}}{\partial\xi^{*\bar\alpha}}\frac{\overrightarrow{\partial}}{\partial\xi^\beta}\right)B. \tag{5.31}$$

The Hamiltonian corresponding to (5.13) is

$$H_A = \tfrac{1}{2}h_{ab}\dot x^a\dot x^b \tag{5.32}$$

$$\dot x^a = h^{ab}[p_b - \tfrac{1}{2}\mathrm{i}\rho_{\alpha\beta}(\Gamma^\alpha{}_{\gamma b}\xi^\gamma\xi^{*\bar\beta} - \Gamma^{*\bar\alpha}{}_{\bar\gamma b}\xi^\beta\xi^{*\bar\gamma})]. \tag{5.33}$$

It can be checked that this Hamiltonian does indeed imply the equations of motion (5.22) and (5.23) using the identity

$$\rho_{\alpha\beta}\Gamma^\beta{}_{\gamma c} = -\rho_{\gamma\beta}\Gamma^{*\bar\beta}{}_{\bar\alpha c} \tag{5.34}$$

which is a consequence of (5.15).

The supercharge Q is given by

$$Q = (p_b + \mathrm{i}\rho_{\alpha\beta}\Delta^{*\bar\alpha}{}_{\bar\gamma b}\xi^\beta\xi^{*\bar\gamma})\xi^b \tag{5.35}$$

$$\Delta^a{}_{bc} = \left\{{a \atop b\ c}\right\} + \mathrm{i}q\gamma^{ad}F_{bc;d}. \tag{5.36}$$

The connection Δ is not compatible with the metric γ_{ab} in the sense of (5.15), but yields (D denoting the corresponding covariant derivative)

$$D_c\gamma_{\bar{a}b} = 2iqF_{ab;c}. \tag{5.37}$$

Incidentally this equation is also useful in the direct verification of $[\xi, Q\} = 0$. We note also that the relation (5.11) is vital for the direct verification of $[Q, Q^*\} = 2iH$.

We now turn to the quantisation of the dynamical system under consideration. As in the gravitational case the occurrence of the metric h_{ab} in (5.13) requires the representation of the quantum states by the tensor densities

$$\tilde{\psi} = |h|^{1/4}\psi \tag{5.38}$$

$$h := \det(h_{ab}) \tag{5.39}$$

and the use of the anholonomic bases introduced in (5.26). Equations (4.19) and (4.20) generalise to

$$\hat{\xi}^\alpha = e^\alpha \wedge \tag{5.40}$$

$$\hat{\xi}^{\dagger\alpha} = e^{\alpha *} \lrcorner. \tag{5.41}$$

As to the correct factor ordering of the quantum version of $\dot{x}^a$, we note that only

$$\hat{\dot{x}}_a = \hat{p}_a + i\rho_{\alpha\beta}\Gamma^{*\bar{\alpha}}{}_{\bar{\gamma}b}\hat{\xi}^\beta\hat{\xi}^{\dagger\gamma} \tag{5.42}$$

operates as a covariant derivative on wavefunctions, namely

$$|h|^{1/4}\hat{\dot{x}}_a|h|^{-1/4}\tilde{\psi} = i\hbar(\nabla^*_a\psi)^\sim \tag{5.43}$$

∇^*_a denoting the covariant derivative with respect to the connection Γ^* (cf 4.25). Likewise the correct factor ordering of the supercharge is

$$\hat{Q}_A = \hat{\xi}^a(|h|^{1/4}\hat{p}_a|h|^{-1/4} + i\rho_{\alpha\beta}\Delta^{*\bar{\alpha}}{}_{\bar{\gamma}a}\xi^\beta\xi^{*\bar{\gamma}}) \tag{5.44}$$

so that

$$\hat{Q}_A = i\hbar^{3/2}|h|^{1/4}\,dx^a \wedge D^*_a|h|^{-1/4} \tag{5.45}$$

$$\hat{Q}^\dagger_A = i\hbar^{3/2}|h|^{1/4}\,D^*_a\,dx^{(a)}\lrcorner|h|^{-1/4} \tag{5.46}$$

where D^*_a denotes the covariant derivative with respect to Δ^* and the brackets surrounding the index a in (5.46) indicate that it has to be associated with the connection $\{\ \}_h$ rather than Δ^* when forming the covariant derivative. For example,

$$\hat{Q}^\dagger_A\tilde{A}^{(1)} = i\hbar^{3/2}|h|^{1/4}\overset{h}{\nabla}_a(\gamma^{ab}A^{(1)}{}_b). \tag{5.47}$$

The mass operator is given by

$$\hat{H}_A = \tfrac{1}{2}|h|^{-1/4}\hat{\dot{x}}_a|h|^{1/2}h^{ab}\hat{\dot{x}}_b|h|^{-1/4} \tag{5.48}$$

$$= -\tfrac{1}{2}\hbar|h|^{1/4}\nabla^*_a h^{(a)b}\nabla^*_b|h|^{-1/4}. \tag{5.49}$$

Note that this operator is self-adjoint because of (5.34) but not real, so that all but the scalar solutions of the mass-shell condition are genuinely complex.

The positivity of the scalar product

$$\langle\psi|\psi\rangle_A = \int d^4x|h|^{1/2}\sum_{p=0}^{4}\frac{1}{p!}A^{(p)}_{i_1\ldots i_p}\gamma^{*i_1j_1}\ldots\gamma^{*i_pj_p}A^{(p)}_{j_1\ldots j_p} \tag{5.50}$$

$$\equiv \int d^4x|h|^{1/2}\psi^*\cdot\psi = \langle\psi|\psi\rangle \tag{5.51}$$

may be shown under the general conditions discussed in § 3. The charge form relevant here is defined by the conserved current implied by the mass-shell condition:

$$j^a(\psi_1, \psi_2) = \mathrm{i}h^{ab}[\psi_1^* \cdot \nabla_b^* \psi_2 - (\nabla_b \psi_1^*) \cdot \psi_2]. \tag{5.52}$$

Both the generalised Lorentz condition $\hat{Q}_A^\dagger \tilde{\psi} = 0$ and the mass-shell condition (for $m^2 > 0$) follow from the variational principle based on the action

$$S[\psi] = -(1/2\hbar^2)(\langle \hat{Q}_A \tilde{\psi} | \hat{Q}_A \tilde{\psi}\rangle + m^2\langle \tilde{\psi} | \tilde{\psi}\rangle). \tag{5.53}$$

The Heisenberg equations of motion reduce in the formal classical limit defined in § 3 to (5.4) and (5.22), respectively, and, for the even spin observables, to

$$\dot{S}^{ab} = 2\gamma^{[b|c} qF_{ic,j} \dot{x}^i T^{|a]j} \tag{5.54}$$

$$\dot{T}^{ab} = 2\gamma^{(a|c} qF_{ic,j} \dot{x}^i S^{j|b)}. \tag{5.55}$$

Formally identical equations are also implied by the pseudoclassical equations (5.5) and (5.23), respectively.

6. Complex Hermitian metric

The results of the two preceding sections can be generalised to the case of an asymmetric metric of the form

$$\gamma_{ab}(x) = g_{ab}(x) + \mathrm{i}B_{ab}(x) \tag{6.1}$$

with both g_{ab} and B_{ab} real and

$$g_{ab} = g_{ba} \qquad B_{ab} = -B_{ba}. \tag{6.2}$$

Note that we are not going to complexify the spacetime manifold itself, but only its tangent bundle. The mathematical aspects of this procedure as well as its prospects as the basis of alternative theories of gravitation have been discussed by Kunstatter and Yates (1981). We expect that analogues of the supersymmtric coupling treated below exist also in the other possible algebraic extensions of the tangent bundle (Kunstatter *et al* 1983, Moffat 1984, Mann 1984).

The supersymmetric Lagrangian is

$$\begin{aligned} L_\gamma &= \tfrac{1}{2} \int \mathrm{d}\theta\, \mathrm{d}\theta^*\, \gamma_{ab}(X) \mathrm{D}X^a \mathrm{D}^* X^b \\ &= \tfrac{1}{2}[g_{ab}(\dot{x}^a \dot{x}^b + y^a y^b) - \mathrm{i}\gamma_{ba}(\xi^a \dot{\xi}^{*b} - \dot{\xi}^a \xi^{*b}) - 2B_{ab}\dot{x}^a y^b \\ &\quad - g_{ab,c}\dot{x}^a S^{bc} - \{cab\}_g y^c T^{ab} + B_{ab,c}\dot{x}^a T^{bc} - \tfrac{3}{2}B_{[ab,c]} y^a S^{bc} \\ &\quad - \tfrac{1}{4} g_{ab,cd} T^{ab} T^{cd} + \tfrac{1}{4} B_{ab,cd} S^{ab} T^{cd}]. \end{aligned} \tag{6.3}$$

We note that this Lagrangian becomes, up to a total derivative, identical to (5.2), if $g_{ab} = \eta_{ab}$ and $B_{ab} = q(A_{b,a} - A_{a,b})$. The underlying reason is that

$$qA_a(X)\dot{X}^a = \tfrac{1}{2}\mathrm{i}qA_a(X)(\mathrm{D}\mathrm{D}^* + \mathrm{D}^*\mathrm{D})X^a$$

yields the same action as

$$\begin{aligned} &-\tfrac{1}{2}\mathrm{i}q[\mathrm{D}A_a(X)\mathrm{D}^*X^a + \mathrm{D}^*A_a(X)\mathrm{D}X^a] \\ &\qquad = -\tfrac{1}{2}\mathrm{i}qA_{a,b}(\mathrm{D}X^b\mathrm{D}^*X^a + \mathrm{D}^*X^b\mathrm{D}X^a) = \tfrac{1}{2}\mathrm{i}q(A_{b,a} - A_{a,b})\mathrm{D}X^a\mathrm{D}^*X^b \end{aligned}$$

owing to the anti-self-adjointness of the supercovariant derivatives with respect to the natural scalar product in superspace.

Substituting the equations of motion for the y variable,

$$g_{ab}y^b = -B_{ab}\dot{x}^b - \tfrac{3}{2}\mathrm{i}B_{[ab,c]}\xi^b\xi^{*c} \tag{6.4}$$

into (6.3) we obtain

$$L_\gamma = \frac{1}{2}\left\{h_{ab}\dot{x}^a\dot{x}^b - i\gamma_{ba}\left[\xi^a\left(\frac{\overset{\Gamma}{\mathrm{D}}\xi^b}{\mathrm{d}s}\right)^* - \frac{\overset{\Gamma}{\mathrm{D}}\xi^a}{\mathrm{d}s}\xi^{*b}\right]\right\} - \tfrac{1}{4}g_{ai}R^i{}_{bcd}(\hat{\Gamma})\xi^a\xi^b\xi^{*c}\xi^{*d} \tag{6.5}$$

$$h_{ab} = g_{ab} + B_{ai}g^{ij}B_{jb} \tag{6.6}$$

$$\hat{\Gamma}^a{}_{bc} = \left\{\begin{matrix} a \\ b\ \ c \end{matrix}\right\}_g + \tfrac{3}{2}\mathrm{i}g^{ad}B_{[db,c]} \tag{6.7}$$

$$\Gamma^a{}_{bc} = \left\{\begin{matrix} a\ \ a \\ b\ \ c \end{matrix}\right\}_g + \mathrm{i}\gamma^{ad}\overset{\hat{\Gamma}}{\nabla}_b B_{\bar{d}\{c\}} \tag{6.8}$$

$$= \left\{\begin{matrix} a \\ b\ \ c \end{matrix}\right\}_g + \gamma^{ad}(\mathrm{i}B_{dc;b} - \tfrac{3}{2}B_{[db,i]}\eta^{ij}B_{jc}). \tag{6.9}$$

We have denoted by $R^i{}_{bcd}(\hat{\Gamma})$ the curvature tensor of the connection $\hat{\Gamma}$. The semicolon in (6.9) denotes the covariant derivative with respect to the Levi-Civita connection $\{\ \}_g$ of g, and the barred and bracketed index in (6.8) are to indicate that $\hat{\Gamma}^*$ and $\{\ \}_g$, respectively, have to appear in the corresponding terms of the covariant derivative. Observe that the connection $\hat{\Gamma}$ is compatible with the metric g, whereas Γ is not. Neither connection is, in general, compatible with the metric γ_{ab}. More precisely we have

$$\overset{\Gamma}{\nabla}_a\gamma_{\bar{b}c} = 3\mathrm{i}B_{[ab,c]}. \tag{6.10}$$

The equations of motion are the following:

$$\ddot{x}^a + \left\{\begin{matrix} a \\ b\ \ c \end{matrix}\right\}_h \dot{x}^b\dot{x}^c = \tfrac{1}{2}\mathrm{i}h^{al}[\gamma_{ci}R^c{}_{jkl}(\Gamma^*)\xi^i\xi^{*j} - \gamma^*_{ci}R^c{}_{jkl}(\Gamma)\xi^j\xi^{*i}]\dot{x}^k$$

$$-\tfrac{1}{2}R^a{}_{bcd}(\hat{\Gamma})\dot{x}^bS^{cd} + \tfrac{1}{4}h^{ab}g_{cd}\overset{\hat{\Gamma}}{\nabla}_b R^c{}_{ijk}(\hat{\Gamma})\xi^d\xi^l\xi^{*j}\xi^{*k} \tag{6.11}$$

$$\frac{\overset{\Gamma}{\mathrm{D}}\xi^a}{\mathrm{d}s} = -i\gamma^{am}g_{ij}R^j{}_{klm}(\tilde{\Gamma})\xi^i\xi^k\xi^{*l} \tag{6.12}$$

$$\tilde{\Gamma}^a{}_{bc} = \left\{\begin{matrix} a \\ b\ \ c \end{matrix}\right\} + \mathrm{i}\gamma^{ad}B_{db;c}. \tag{6.13}$$

We now introduce a complex orthonormal vierbein in the same manner as indicated in (5.26) with formally the same consequences as expressed in (5.29)-(5.31). The Hamiltonian is then

$$H_\gamma = \tfrac{1}{2}h_{ab}\dot{x}^a\dot{x}^b - \tfrac{1}{4}g_{ai}R^i{}_{bcd}(\hat{\Gamma})\xi^a\xi^{*c}\xi^b\xi^{*d} \tag{6.14}$$

where $\dot{x}$ has the same formal appearance as in (5.33), with Γ given by (6.8). The supercharges are given by a remarkably simple expression, namely

$$Q_\gamma = (p_a + \mathrm{i}\rho_{\alpha\beta}\Delta^{*\bar{\alpha}}{}_{\bar{\gamma}a}\xi^\beta\xi^{*\bar{\gamma}})\xi^a \tag{6.15}$$

$$\Delta^a{}_{bc} = \left\{\begin{matrix} a \\ b\ \ c \end{matrix}\right\}_g + \mathrm{i}\gamma^{ad}B_{bc;d} \tag{6.16}$$

(cf (5.35) and (5.36)). The non-metricity tensor associated with the connection Δ is

$$D_c\gamma_{\bar{a}b} = \mathrm{i}(B_{ab;c} + B_{ac;b} + B_{cb;a}). \tag{6.17}$$

Most of the quantum theoretical discussion of the preceding section applies also to the present case. The formulae (5.38)-(5.47) and (5.50)-(5.53) may be taken over virtually unchanged if interpreted in the appropriate way.

The general structure of the mass operator is

$$\hat{H}_B = \tfrac{1}{2}|h|^{-1/4}\hat{x}_a|h|^{1/2}h^{ab}\hat{x}_b|h|^{-1/4} + \hat{R} \tag{6.18}$$

$$= -\tfrac{1}{2}\hbar^2(|h|^{1/4}\nabla^*_a h^{(a)b}\nabla^*_b|h|^{-1/4} + \hat{R} \tag{6.19}$$

$$\hat{R} = \ -\tfrac{1}{4}[g_{ai}R^i{}_{bcd}(\hat{\Gamma})\xi^a\xi^{*c}\xi^b\xi^{*d}]^\wedge. \tag{6.20}$$

The exact factor ordering of $\hat{R}$ is rather intricate, as (4.32) generalises to

$$\hat{R}^a{}_{[bcd]} = -\mathrm{i}\overset{\hat{\Gamma}}{\nabla}_{[b}f^a{}_{cd]} + 2f^i{}_{[bc}f^a{}_{d]i} \tag{6.21}$$

$$f^a{}_{bc} \equiv 3g^{ad}B_{[db,c]}. \tag{6.22}$$

Therefore $g_{ai}R^i{}_{bcd}(\hat{\Gamma})\xi^a\xi^{*c}\xi^b\xi^{*d}$ is different from $-\tfrac{1}{2}g_{ai}R^i{}_{bcd}(\hat{\Gamma})S^{ab}S^{cd}$ even at the classical level. Note however that the operator versions of both expressions are self-adjoint, because the curvature tensor is Hermitian in the sense that

$$g_{ai}R^i{}_{bcd}(\hat{\Gamma}) = g_{ci}R^i{}_{dab}(\hat{\Gamma}^*). \tag{6.23}$$

Because of the complicated structure of $\hat{R}$ we forego the computation of the explicit form of the Heisenberg equations of motion and their classical limit.

7. Discussion

The main results of this paper can be summarised as follows. First, free massive spin-1 and spin-0 particles, realised by antisymmetric tensor fields, are the maximally (i.e. $N = 1$) supersymmetric states of definite 'fermion' number in relativistic quantum mechanics with $N = 2$ proper-time supersymmetry. Massless helicity-0 and helicity-1 particles are even $N = 2$ supersymmetric, with one of the supersymmetries corresponding to the Abelian gauge invariance of an antisymmetric tensor field representation. Note that higher rank antisymmetric tensor gauge fields appear naturally in supergravity and string field theories and are serious candidates to number among the fundamental fields, as their classical equivalence to the standard representations does not survive at the second quantised level (Duff and van Nieuwenhuizen 1980). We find it remarkable that these fields, as well as the Dirac field, possess an underlying pseudoclassical particle dynamics. In this context it is appropriate to mention that there exist 'superparticle' models corresponding to general Poincaré (Balachandran *et al* 1983) and super-Poincaré covariant field theories (Stern 1985, Green and Schwarz 1984). However these approaches differ significantly from the present one in that they employ concepts of spacetime (super) symmetry like representation matrices, spinors, etc, whereas we are dealing only with a classical position coordinate and its anticommuting counterpart.

The second remarkable fact is the severe restriction imposed by supersymmetry on the possible couplings of the particles and, on the other hand, the richness of geometrical structure of the bosonic configuration space that is implied by these couplings. Maybe the most surprising result is that the minimal electromagnetic coupling is not compatible with $N=2$ supersymmetry. This result anticipates the inconsistency of the minimal electromagnetic coupling of gauge fields (Arnowitt and Deser 1963) and is indeed confirmed at the second quantised level by a no-go theorem of Weinberg and Witten (1980), whose proof was completed by Lopuszánski (1984): a massless particle of helicity $|h|>\frac{1}{2}$ cannot carry a charge of an internal symmetry induced by a Lorentz-covariant conserved current. There is also no experiment that would contradict the vanishing of the electric charge of vector particles. Note that the weak gauge bosons W^+ and W^- belong to a non-Abelian gauge triplet of vector fields and that their electromagnetic interaction is part of the self-interaction of this triplet. This self-interaction does of course not show up in the linear field theory obtained by the quantisation of classical particles. $N=2$ supersymmetry forbids also the minimal coupling to torsion in a Riemann-Cartan spacetime, in contrast to the Dirac case (Rumpf 1982). On the other hand, the couplings to scalar fields and complex Hermitian tensor fields are peculiar to $N=2$ supersymmetry.

The third type of interesting information is contained in the classical limit of the Heisenberg equations of motion for the position and spin observables of the particle associated with antisymmetric tensor fields. This classical limit is formulated in terms of real rather than Grassmann numbers and has to be distinguished conceptually from the underlying pseudoclassical mechanics. We feel that the results obtained are relevant for the analysis of particle motion in external fields, provided the latter are not so strong as to make the particle concept ill-defined. As an example we quote the apparent success of the Bargmann-Michel-Telegdi equation (Bargmann *et al* 1959). Experimental evidence suggests that all fundamental $N=2$ supersymmetric particles are massless. Since they possess only two independent polarisations (in the case of helicity $|h|>0$) they will not exhibit the full complexity of possible spin motions allowed by the classical limit equations.

We close with some speculations about generalisations of the approach adopted in the present work. First, on the basis of the results of $N=1$ and $N=2$ proper-time supersymmetry, it is natural to conjecture that spin (helicity) s states will occur in $N=2s$ supersymmetric relativistic quantum mechanics. Second, it may be possible to characterise quantised supersymmetric *field* theories by the geometric structure of the space of bosonic field configurations and the 'external field' couplings implied by this structure for the wavefunctional on this space. The relevance of geometrical concepts in field configuration space even for non-supersymmetric quantum field theory has been demonstrated recently (Vilkovisky 1984, Rumpf 1986b). The sought-after characterisation would imply 'supersymmetry without anticommuting variables' and thus have a status comparable to that of the definition of the Nicolai map (Nicolai 1980a, b).

Appendix. Non-viability of operator formalism for minimal electromagnetic coupling

The non-minimal character of the supersymmetric coupling to a vector field raises the question whether there exists a classical particle dynamics underlying the minimally coupled antisymmetric tensor fields. In the following we shall argue that this is not the case. We consider the minimally coupled Proca field ψ^c with Lagrangian (Wentzel 1949)

$$\mathscr{L}'_A=-\tfrac{1}{4}f^*_{ab}f_{ab}+(m^2/2\hbar^2)\psi^*_c\psi^c \tag{A1}$$

$$f_{ab}=(\partial_a+\mathrm{i}eA_a)\psi_b-(\partial_b+\mathrm{i}eA_b)\psi_a. \tag{A2}$$

The resulting field equation

$$(\partial_a + ieA_a)f^{ab} + (m^2/\hbar^2)\psi^b = 0 \tag{A3}$$

does not simplify much if its own consequence

$$(\partial_a + ieA_a)\psi^a = \tfrac{1}{2}i(e\hbar^2/m^2)F_{ab}f^{ab} \tag{A4}$$

is used. This is the main obstacle for the construction of a particle dynamics. The first step in this construction would be the definition of a 'quantum Hamiltonian' $\hat{H}$. One would try to identify $\hat{H}$ with part of the operator $\hat{H}'$ defined by the classical action $S[\psi]$ via

$$\hbar^2 S[\psi] = \langle\psi| - \hat{H}' + (m^2/2)|\psi\rangle. \tag{A5}$$

In the case of the Lagrangian (A1) we have

$$\hat{H}' = \tfrac{1}{2}[(\hat{p} - eA)^2 - (\hat{p}_a - eA_a)(\hat{p}_b - eA_b)e^a e^{b\dagger}] \tag{A6}$$

$$= \tfrac{1}{2}[(\hat{p} - eA)^2 - \tfrac{1}{4}\{\hat{p}_a - eA_a, \hat{p}_b - eA_b\}e^{(aeb)^\dagger} - \tfrac{1}{2}eF_{ab}\hat{S}^{ab}] \tag{A7}$$

where e^a are standard basis vectors and the dagger denotes Hermitian conjugation. If the right-hand side of (A4) were zero, we could discard the second term in the square bracket in (A7). The remainder is easily recognised as the Hamiltonian describing a charged spinning particle with gyromagnetic ratio $g = 1$. This follows from the Heisenberg equations of motion for $\hat{x}$, $\hat{p}$, $\hat{S}_{ab}$. (We have not addressed the question whether a pseudoclassical Lagrangian could be found yielding this quantum Hamiltonian.) Indeed it is well known (Wentzel 1949) that $g = 1$ for the minimally coupled Proca field, and (A7) is presumably the shortest 'derivation' of this result. But the right-hand side of (A4) is not zero, and hence no sensible Heisenberg equations of motion for the particle observables can be derived.

A considerable simplification occurs if the following non-minimal coupling is introduced. Consider

$$\mathscr{L}''_A = \mathscr{L}'_A - (e/2\hbar)F_{ab}\psi^{a*}\psi^b. \tag{A8}$$

This results in

$$\hat{H}'' = \hat{H}' - \tfrac{1}{4}eF_{ab}\hat{S}^{ab} \tag{A9}$$

and hence $g = 2$. This modification and its consequences for second quantisation have been studied in detail by Lee and Yang (1962). In our context (which is to exhibit the classical field equation as a *first* quantised theory) an interesting consequence of the modification is

$$\hat{H}''(p_a - eA_a)e^a = \tfrac{1}{2}e\hbar F^{ac}{}_{,c}e_a. \tag{A10}$$

The right-hand side of (A10) vanishes *in vacuo* ($F^{ac}{}_{,c} = 0$). In this case one may conclude

$$\psi^\dagger e^a(\hat{p}_a - eA_a) = 0 \tag{A11}$$

if ψ satisfies $\hat{H}''\psi = \frac{1}{2}m^2\psi$. Thus (A11) may be considered as a subsidiary condition on the space of solutions of the classical field equation. Hence on this space

$$\hat{H}'' = \tfrac{1}{2}[(\hat{p} - eA)^2 - e\hbar F_{ab}\hat{S}^{ab}]. \tag{A12}$$

This yields the Heisenberg equations of motion

$$\ddot{\hat{x}}^a = \tfrac{1}{2}e\{F^a{}_b, \dot{\hat{x}}^b\} + \tfrac{1}{2}e\hbar\eta^{ad}F_{bc,d}\hat{S}^{bc} \tag{A13}$$

$$\dot{\hat{S}}^{ab} = e(F^a{}_c\hat{S}^{cb} + F^b{}_c\hat{S}^{ac}). \tag{A14}$$

In the classical limit these equations are very familiar: (A13) describes obviously the translational motion of a $g=2$ particle. Equation (A14) is formally identical with the spin equation of motion obtained from the Dirac equation. In the case of constant F_{ab} it is a special case of the Bargmann–Michel–Telegdi equation (Bargmann *et al* 1959). However this derivation works only *in vacuo* and on mass shell, and a pseudo-classical Lagrangian for H'' involving the appropriate degrees of freedom apparently does not exist (although (A12) is formally identical with the Dirac particle Hamiltonian (Rumpf 1982)).

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